Subdomain Perturbation Finite Element Method for Skin and Proximity Effects

Patrick Dular, Ruth V. Sabariego and Laurent Krähenbühl

Abstract—**Skin and proximity effects are calculated in both active and passive conductors via a subproblem finite element method based on a perturbation technique. A reference limit problem is first solved by considering either perfect conductors or magnetic materials via appropriate boundary conditions. Its solution gives then the source for the eddy current perturbation subproblems in each conductor with its actual conductivity or permeability and its own mesh. The proposed method accurately determines the current density distributions and ensuing losses in conductors of any shape in both frequency and time domains, which overcomes the limitations of the impedance boundary condition technique.**

Index Terms—**Eddy currents, finite element method, perturbation method, skin and proximity effects.**

I. Introduction

A precise consideration of the skin and proximity effects in active and passive conductors is important for an accurate calculation of their field distribution and the ensuing Joule losses. Calculating these effects with the classical finite element (FE) method usually presents difficulties. The mesh must be fine enough with respect to the skin depth in all the materials, which then leads to a large system of equations.

Impedance boundary conditions (IBCs) [1] defined on the conductor boundaries are an alternative to avoid meshing the conductor interior. Such boundary conditions (BCs) are nevertheless generally based on analytical solutions and in practice only valid far from any geometrical discontinuities, e.g. edges and corners. They are also generally restricted to frequency domain and linear analyses.

In this contribution, a method is developed to overcome the limitations of IBCs, considering conductors of any shape not only in the frequency domain but also in the time domain. The magnetic vector potential FE magnetodynamic formulation is used. The developed method is based on the coupling of reference and perturbation solutions [2]-[4], each of these being calculated in distinct meshes. A reference limit eddy current FE problem is first solved by considering either perfect conductive or magnetic properties, via appropriate BCs on the conductor boundaries. The solution of the limit problem gives the sources for FE perturbation subproblems in each conductor then considered with a finite conductivity or permeability. Each of these problems requires an appropriate volume mesh of the associated conductor and its surrounding region. The developed technique is validated on an application example. Its main advantages versus the IBC technique are pointed out.

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II. Reference and Modified Eddy Current Problems

A. Reference problem and its strong formulation

A given problem p consists in solving the magnetodynamic equations in a bounded domain Ω_p, with boundary $\Gamma_p = \Gamma_{h,p} \cup \Gamma_{b,p} = \partial\Omega_p$ (possibly at infinity), of the 2-D or 3-D Euclidean space. The eddy current conducting part of Ω_p is denoted $\Omega_{c,p}$ and the non-conducting one $\Omega_{c,p}^{C}$, with $\Omega_p = \Omega_{c,p} \cup \Omega_{c,p}^{C}$. Massive conductors belong to $\Omega_{c,p}$.

A problem, defined as problem $p = 1$, is first considered. Its equations and material relations in Ω_1, and boundary conditions (BCs) and interface conditions (ICs), are

$$\operatorname{curl} \boldsymbol{h}_1 = \boldsymbol{j}_1 \,, \quad \operatorname{curl} \boldsymbol{e}_1 = -\partial_t \boldsymbol{b}_1 \,, \quad \operatorname{div} \boldsymbol{b}_1 = 0 \,, \qquad \text{(1a-b-c)}$$

$$\boldsymbol{b}_1 = \mu_1 \boldsymbol{h}_1 \,, \quad \boldsymbol{j}_1 = \sigma_1 \boldsymbol{e}_1 \,, \qquad \text{(1d-e)}$$

$$\boldsymbol{n} \times \boldsymbol{h}_1|_{\Gamma_{h,1}} = 0 \,, \ \boldsymbol{n} \times \boldsymbol{e}_1|_{\Gamma_{e,1} \subset \Gamma_{b,1}} = 0 \,, \ \boldsymbol{n} \cdot \boldsymbol{b}_1|_{\Gamma_{b,1}} = 0 \,, \qquad \text{(1f-g-h)}$$

$$[\boldsymbol{n} \times \boldsymbol{h}_1]_{\gamma_1} = \boldsymbol{j}_{su,1}, \ [\boldsymbol{n} \times \boldsymbol{e}_1]_{\gamma_1} = \boldsymbol{k}_{su,1}, \ [\boldsymbol{n} \cdot \boldsymbol{b}_1]_{\gamma_1} = \boldsymbol{b}_{su,1}, \qquad \text{(1i-j-k)}$$

where $\boldsymbol{h}$ is the magnetic field, $\boldsymbol{b}$ is the magnetic flux density, $\boldsymbol{e}$ is the electric field, $\boldsymbol{j}$ is the electric current density (including source and eddy currents), μ is the magnetic permeability, σ is the electric conductivity and $\boldsymbol{n}$ is the external unit normal to a boundary. Note that (1b) is only expressed in $\Omega_{c,1}$, whereas it is reduced to the form (1c) in $\Omega_{c,1}^{C}$. Also (1g) is more restrictive than (1h). The notation $[\,\cdot\,]_\gamma = \cdot|_{\gamma^+} - \cdot|_{\gamma^-}$ expresses the discontinuity of a quantity through any interface γ (of both sides γ^+ and γ^-), which is allowed to be non-zero (the associated surface fields $\boldsymbol{j}_{su,1}$, $\boldsymbol{k}_{su,1}$ and $\boldsymbol{b}_{su,1}$ are usually unknown, i.e., parts of the solution). The subscript of each quantity refers to the associated problem p. As it will be shown, it is intended to solve successive problems, the solutions of which being added to get the solution of a complete problem. At this first step, the solution $p = 1$ is a solution as a whole, called reference or source solution.

B. Modified problem defining perturbations

A modification of the problem $p = 1$ due to a change of permeability or conductivity in some subregions leads to a perturbation of each field quantity. Both large and small perturbations are considered. This can result from the change of properties of existing materials [2] or from the addition of new materials in the ambient region [3], which actually also amounts to changing some material properties. The governing equations and relations in another domain Ω_2, i.e. a modified form of Ω_1, and the BCs and ICs, are still of the form (1) with all the involved quantities relative to the new solution and the involved boundaries relative to Ω_2 (also with the subscript 2). To point out the decomposition of the new solution as a perturbation of the reference one, these quantities are written as

$$\boldsymbol{h}=\boldsymbol{h}_1+\boldsymbol{h}_2\,,\ \boldsymbol{b}=\boldsymbol{b}_1+\boldsymbol{b}_2\,,\ \boldsymbol{j}=\boldsymbol{j}_1+\boldsymbol{j}_2\,,\ \boldsymbol{e}=\boldsymbol{e}_1+\boldsymbol{e}_2\,,$$
$$\boldsymbol{j}_{su}=\boldsymbol{j}_{su,1}+\boldsymbol{j}_{su,2}\,,\ \boldsymbol{k}_{su}=\boldsymbol{k}_{su,1}+\boldsymbol{k}_{su,2}\,,\ \boldsymbol{b}_{su}=\boldsymbol{b}_{su,1}+\boldsymbol{b}_{su,2}\,. \qquad (2)$$

Subtracting each equation of (1) from its counterpart in the so-modified problem, the perturbation equations that define the problem $p=2$, are

$$\operatorname{curl}\boldsymbol{h}_2=\boldsymbol{j}_2\,,\quad \operatorname{curl}\boldsymbol{e}_2=-\partial_t\boldsymbol{b}_2\,,\quad \operatorname{div}\boldsymbol{b}_2=0\,, \qquad \text{(3a-b-c)}$$
$$\boldsymbol{b}_2=\mu_2\,\boldsymbol{h}_2+\boldsymbol{b}_{s,2}\,,\quad \boldsymbol{j}_2=\sigma_2\,\boldsymbol{e}_2+\boldsymbol{j}_{s,2}\,, \qquad \text{(3d-e)}$$
$$\boldsymbol{n}\times\boldsymbol{h}_2|_{\Gamma_{h,2}}=0\,,\ \boldsymbol{n}\times\boldsymbol{e}_2|_{\Gamma_{e,2}\subset\Gamma_{b,2}}=0\,,\ \boldsymbol{n}\cdot\boldsymbol{b}_2|_{\Gamma_{b,2}}=0\,, \qquad \text{(3f-g-h)}$$
$$[\boldsymbol{n}\times\boldsymbol{h}_2]_{\gamma_2}=\boldsymbol{j}_{su,2}\,,\ [\boldsymbol{n}\times\boldsymbol{e}_2]_{\gamma_2}=\boldsymbol{k}_{su,2}\,,\ [\boldsymbol{n}\cdot\boldsymbol{b}_2]_{\gamma_2}=\boldsymbol{b}_{su,2}\,, \qquad \text{(3i-j-k)}$$

where the so-defined volume sources $\boldsymbol{b}_{s,2}$ and $\boldsymbol{j}_{s,2}$ are obtained from the reference solution as

$$\boldsymbol{b}_{s,2}=(\mu_2-\mu_1)\,\boldsymbol{h}_1\,,\quad \boldsymbol{j}_{s,2}=(\sigma_2-\sigma_1)\,\boldsymbol{e}_1\,. \qquad \text{(4-5)}$$

The perturbation fields are still governed by the Maxwell equations in their classical forms (3a-b-c) whereas their associated material relations now include additional sources (4) and (5). These sources only occur in the modified regions.

Solving the perturbation problem (3) instead of the modified one enables to avoid operations already performed in the reference problem (1). At the discrete level, the meshes of both reference and perturbation problems can be significantly simplified, each problem asking for mesh refinement of different regions.

By construction, the summation of the solutions of (1) and (3), i.e. (2), gives exactly the solution of the modified problem under the condition that all the materials are linear and that the wished changes in domain Ω_2 with reference to domain Ω_1 are expressed by (4) and (5) in (3d-e).

C. Possible approximations in the perturbation problem

For the sake of simplicity at the discrete level, domain Ω_2 can neglect some materials initially present in domain Ω_1, while these must be present in the considered complete problem. Any intersection of non-material regions of Ω_2 with the material regions of Ω_1 is thus allowed.

The corrections (4) and (5) are therefore not applicable in the omitted material regions. The material relations for the complete fields in these regions are thus approximated as

$$\boldsymbol{b}_1+\boldsymbol{b}_2=\mu_1\,\boldsymbol{h}_1+\mu_2\,\boldsymbol{h}_2\,,\quad \boldsymbol{j}_1+\boldsymbol{j}_2=\sigma_1\,\boldsymbol{e}_1+\sigma_2\,\boldsymbol{e}_2\,, \qquad \text{(6a-b)}$$

which result from the summation of (1d-e) with their counterparts in (3d-e) without volume sources (4) and (5). These relations can be transformed, with (2), as

$$\boldsymbol{b}=\mu_1\,\boldsymbol{h}+(\mu_2-\mu_1)\,\boldsymbol{h}_2\,,\quad \boldsymbol{j}=\sigma_1\,\boldsymbol{e}+(\sigma_2-\sigma_1)\,\boldsymbol{e}_2\,, \qquad \text{(7a-b)}$$

to point out the error made when their second terms are not negligible, which is the case when the material properties differ too much and the perturbation fields are too large compared to the reference fields. The correct relations, $\boldsymbol{b}=\mu_1\,\boldsymbol{h}$ and $\boldsymbol{j}=\sigma_1\,\boldsymbol{e}$, only occur in the regions where μ and σ are unchanged.

For large perturbations, an iterative procedure between the problems is required to obtain an accurate solution. Each subproblem is responsible for giving the suitable correction, as a perturbation. Such iterations also allow nonlinear analyses.

III. The Reference Problem as a Limit Case

A. Reference problem with perfect conductors

A first reference problem considers conductors $\Omega_{cpe,1}\subset\Omega_{c,1}\subset\Omega_1$, of boundary $\Gamma_{cpe,1}=\partial\Omega_{cpe,1}$, as perfect, i.e. with $\sigma_1\to\infty$. This results in a zero skin depth and thus surface currents. The surface currents are considered to flow between $\Gamma_{cpe,1}^{+}$ and $\Gamma_{cpe,1}^{-}$, the inner and outer sides of $\Gamma_{cpe,1}$ with regard to $\Omega_{cpe,1}$. The domain $\Omega_{cpe,1}$ can be extracted from Ω_1 in (1) and treated via a BC of zero normal magnetic flux density on its boundary $\Gamma_{cpe,1}^{+}$. Given that only zero fields exist in $\Omega_{cpe,1}$, the same BC appears on $\Gamma_{cpe,1}^{-}$. One thus has

$$\boldsymbol{n}\cdot\boldsymbol{b}_1|_{\Gamma_{cpe,1}^{+}}=0\,,\quad \boldsymbol{n}\cdot\boldsymbol{b}_1|_{\Gamma_{cpe,1}^{-}}=0\,. \qquad \text{(8a-b)}$$

Further, the trace of the magnetic field is unknown on $\Gamma_{cpe,1}^{+}$ and vanishes on $\Gamma_{cpe,1}^{-}$, i.e. with (1i),

$$\boldsymbol{n}\times\boldsymbol{h}_1|_{\Gamma_{cpe,1}^{+}}=\boldsymbol{j}_{su,1}\,,\quad \boldsymbol{n}\times\boldsymbol{h}_1|_{\Gamma_{cpe,1}^{-}}=0\,. \qquad \text{(9a-b)}$$

The modified problem (3) now considers $\Omega_{cpe,2}=\Omega_{cpe,1}\subset\Omega_{c,2}$ with its finite conductivity σ_2, which alters the distribution of the eddy current density and the other fields. The fields in $\Omega_{cpe,2}$ are not surface fields anymore but penetrate the conductors. They are solutions of problem (3) with Ω_2 now including $\Omega_{cpe,2}$ but with particular ICs (3i-k) through $\Gamma_{cpe,2}=\partial\Omega_{cpe,2}$.

On the one hand, (3k) with (2) and (1k) leads to

$$[\boldsymbol{n}\cdot\boldsymbol{b}_2]_{\Gamma_{cpe,2}}=\boldsymbol{b}_{su,2}=\boldsymbol{b}_{su}-\boldsymbol{b}_{su,1}=[\boldsymbol{n}\cdot\boldsymbol{b}]_{\Gamma_{cpe,2}}-\boldsymbol{b}_{su,1}=0, \qquad (10)$$

due to the continuity of $\boldsymbol{n}\cdot\boldsymbol{b}$ in the complete solution (2) and the zero value of $\boldsymbol{b}_{su,1}$ via (8a-b).

On the other hand, (3i) with (2) and (1i) leads to

$$[\boldsymbol{n}\times\boldsymbol{h}_2]_{\Gamma_{cpe,2}}=\boldsymbol{j}_{su,2}=[\boldsymbol{n}\times\boldsymbol{h}]_{\Gamma_{cpe,2}}-\boldsymbol{j}_{su,1}=-\boldsymbol{n}\times\boldsymbol{h}_1|_{\Gamma_{cpe,1}^{+}}, \qquad (11)$$

due to the continuity of $\boldsymbol{n}\times\boldsymbol{h}$ in (2) and relation (9a).

B. Reference problem with perfect magnetic materials

Another reference problem considers now conductors $\Omega_{cpm,1}\subset\Omega_{c,1}\subset\Omega_1$, of boundary $\Gamma_{cpm,1}=\partial\Omega_{cpm,1}$, as perfect magnetic materials, i.e. with $\mu_1\to\infty$. The domain $\Omega_{cpm,1}$ can thus be extracted from Ω_1 in (1) and treated via a BC fixing a zero tangential magnetic field on its boundary $\Gamma_{cpm,1}^{+}$. Because only zero fields exist in $\Omega_{cpm,1}$, the same BC appear on $\Gamma_{cpm,1}^{-}$. One thus has

$$\boldsymbol{n}\times\boldsymbol{h}_1|_{\Gamma_{cpm,1}^{+}}=0\,,\quad \boldsymbol{n}\times\boldsymbol{h}_1|_{\Gamma_{cpm,1}^{-}}=0\,. \qquad \text{(12a-b)}$$

Also, the trace of the magnetic flux density is unknown on $\Gamma_{cpm,1}^{+}$ and vanishes on $\Gamma_{cpm,1}^{-}$, i.e. with (1k),

$$\boldsymbol{n}\cdot\boldsymbol{b}_1|_{\Gamma_{cpm,1}^{+}}=\boldsymbol{b}_{su,1}\,,\quad \boldsymbol{n}\cdot\boldsymbol{b}_1|_{\Gamma_{cpm,1}^{-}}=0\,. \qquad \text{(13a-b)}$$

The modified problem (3) now considers $\Omega_{cpm,2}=\Omega_{cpm,1}$ with its finite permeability μ_2. The perturbation fields are solutions of problem (3) with Ω_2 now including $\Omega_{cpm,2}$ but with particular ICs (3i-k) through $\Gamma_{cpm,2}=\partial\Omega_{cpm,2}$.

On the one hand, (3k) with (2) and (1k) leads to

$$[\boldsymbol{n}\cdot\boldsymbol{b}_2]_{\Gamma_{cpm,2}}=\boldsymbol{b}_{su,2}=[\boldsymbol{n}\cdot\boldsymbol{b}]_{\Gamma_{cpm,2}}-\boldsymbol{b}_{su,1}=-\boldsymbol{n}\cdot\boldsymbol{b}_1|_{\Gamma_{cpm,1}^{+}}, \qquad (14)$$

due to the continuity of $\boldsymbol{n}\cdot\boldsymbol{b}$ in (2) and relation (13a).

On the other hand, (3i) with (2) and (1i) leads to

$$[\boldsymbol{n}\times\boldsymbol{h}_2]_{\Gamma_{cpm,2}}=\boldsymbol{j}_{su,2}=[\boldsymbol{n}\times\boldsymbol{h}]_{\Gamma_{cpm,2}}-\boldsymbol{j}_{su,1}=0, \qquad (15)$$

due to the continuity of $\boldsymbol{n}\times\boldsymbol{h}$ in (2) and $\boldsymbol{j}_{su,1}=0$ via (12a-b).

IV. Finite Element Weak Formulations

A. *b*-conform weak magnetodynamic formulations

The eddy current problems p are defined in Ω_p with the magnetic vector potential formulation [5], expressing the electric field $\boldsymbol{e}_p$ in $\Omega_{c,p}$ via a magnetic vector potential $\boldsymbol{a}_p$ together with the gradient of an electric scalar potential v_p, and the magnetic flux density $\boldsymbol{b}_p$ in Ω_p as the curl of $\boldsymbol{a}_p$.

The resulting $\boldsymbol{a}$-v magnetodynamic formulation of problem $p=1$ is obtained from the weak form of the Ampère equation (1a), i.e. [5],

$$(\mu_1^{-1} \operatorname{curl} \boldsymbol{a}_1, \operatorname{curl} \boldsymbol{a}')_{\Omega_1} + (\sigma_1 \partial_t \boldsymbol{a}_1, \boldsymbol{a}')_{\Omega_{c,1}} + (\sigma_1 \operatorname{grad} v_1, \boldsymbol{a}')_{\Omega_{c,1}} + <\boldsymbol{n}\times\boldsymbol{h}_1, \boldsymbol{a}'>_{\Gamma_{h,1}} + <\boldsymbol{n}\times\boldsymbol{h}_1, \boldsymbol{a}'>_{\Gamma_{b,1}} = 0, \ \forall \boldsymbol{a}' \in F_1^1(\Omega_1), \quad (16)$$

where $F_1^1(\Omega_1)$ is a gauged curl-conform function space defined on Ω_1 and containing the basis functions for $\boldsymbol{a}$ as well as for the test function $\boldsymbol{a}'$ (at the discrete level, this space is defined by edge finite elements); $(\cdot\,,\cdot)_\Omega$ and $<\cdot\,,\cdot>_\Gamma$ respectively denote a volume integral in Ω and a surface integral on Γ of the product of their vector field arguments. The surface integral term on $\Gamma_{h,1}$ accounts for natural BCs of type (1f). The term on the surface $\Gamma_{b,1}$ with essential BCs on $\boldsymbol{n}\cdot\boldsymbol{b}_1$ is usually omitted because it gives no local contribution to (16). It will be shown to be the key for the post-processing of the reference solution, a part of which being $\boldsymbol{n}\times\boldsymbol{h}_1|_{\Gamma_{b,1}}$.

The weak formulation of the perturbation problem (3) is

$$(\mu_2^{-1} \operatorname{curl} \boldsymbol{a}_2, \operatorname{curl} \boldsymbol{a}')_{\Omega_2} + (\sigma_2 \partial_t \boldsymbol{a}_2, \boldsymbol{a}')_{\Omega_{c,2}\setminus\Omega_{cp,2}} + (\sigma_2 \operatorname{grad} v_2, \boldsymbol{a}')_{\Omega_{c,2}\setminus\Omega_{cp,2}} + (\sigma_2 \partial_t \boldsymbol{a}_2, \boldsymbol{a}')_{\Omega_{cp,2}} + (\sigma_2 \operatorname{grad} v_2, \boldsymbol{a}')_{\Omega_{cp,2}} + <\boldsymbol{n}\times\boldsymbol{h}_2, \boldsymbol{a}'>_{\Gamma_{h,2}} + <[\boldsymbol{n}\times\boldsymbol{h}_2]_{\Gamma_{cp,2}}, \boldsymbol{a}'>_{\Gamma_{cp,2}} = 0, \ \forall \boldsymbol{a}' \in F_2^1(\Omega_2), \quad (17)$$

where $\Omega_{cp,2}$ stands for the perturbed conducting region, i.e. either $\Omega_{cpe,2}$ or $\Omega_{cpm,2}$. A major consequence of the $\boldsymbol{b}$-conform formulation used is that ICs (10) and (11) are to be defined respectively in strong and weak senses, i.e. (10) in $F_2^1(\Omega_2)$ and (11) in a surface integral term.

Note that the approximation (6b) amounts to neglecting the second and third terms of (17), while (6a) acts on its first term.

B. Perfect conductors perturbed to real ones

For a perfect conductor domain $\Omega_{cpe,1}$, the BC (8a) leads to an essential BC on the primary unknown $\boldsymbol{a}_1$ that can be expressed via the definition of a surface scalar potential u_1 (in general single valued, if no net magnetic flux flows in $\Omega_{cpe,1}$) [6], i.e.,

$$\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{a}_1|_{\Gamma_{cpe,1}} = 0 \Leftrightarrow \boldsymbol{n}\times\boldsymbol{a}_1|_{\Gamma_{cpe,1}} = \boldsymbol{n}\times\operatorname{grad} u_1|_{\Gamma_{cpe,1}}. \quad (18)$$

In a 2D model with perpendicular currents, condition (18) amounts to defining a floating (constant) value for the perpendicular component of $\boldsymbol{a}_1$ for each conductor.

The reference formulation is of the form (16) where the perfect conductors are extracted from Ω_1 and $\Omega_{c,1}$ and are only involved through their boundaries $\Gamma_{cpe,1}$ with the condition (18). This latter condition is to be strongly defined in $F_1^1(\Omega_1)$. The boundaries $\Gamma_{cpe,1}$ are thus to be added to $\Gamma_{b,1}$.

The surface integral term $<\boldsymbol{n}\times\boldsymbol{h}_1, \boldsymbol{a}'>_{\Gamma_{cpe,1}}$ is non-zero only for the function grad u' (from (18)), the value of which is then the total surface current I_1 flowing in $\Gamma_{cpe,1}$ (this can be demonstrated from the general procedure developed in [6]). It is zero for all the other local test functions (at the discrete level, for any edge not belonging to $\Gamma_{cpe,1}$). This way, the circuit relation can be expressed for each conductor $\Omega_{cpe,1}$ and the coupling with electrical circuits is possible.

For the associated perturbation formulation (17), the IC (10) is strongly expressed via the continuity of the vector potential $\boldsymbol{a}_2$ through $\Gamma_{cpe,2}$. The IC (11) can rather only act in a weak sense via the surface integral term related to $\Gamma_{cpe,2}^+$ in (17). Indeed, its involved quantity $\boldsymbol{n}\times\boldsymbol{h}_1$ is not known in a strong sense on $\Gamma_{cpe,2}^+$, but rather in a weak sense. One has

$$<[\boldsymbol{n}\times\boldsymbol{h}_2]_{\Gamma_{cpe,2}}, \boldsymbol{a}'>_{\Gamma_{cpe,2}} = <-\boldsymbol{n}\times\boldsymbol{h}_1, \boldsymbol{a}'>_{\Gamma_{cpe,2}^+} = <-\boldsymbol{n}\times\boldsymbol{h}_1, \boldsymbol{a}'>_{\Gamma_{cpe,1}^+} = -(\mu_1^{-1}\operatorname{curl}\boldsymbol{a}_1, \operatorname{curl}\boldsymbol{a}')_{\Omega_1\setminus\Omega_{cpe,1}} \quad (19)$$

in case no part of $\Omega_{c,1}\setminus\Omega_{cpe,1}$ is in contact with $\Omega_{cpe,1}$ (otherwise the second and third terms of (16) have to be considered as well). This way, the surface integral term related to $\Gamma_{cpe,2}^+$ in (17) is calculated from a volume integral coming from the reference problem. Its consideration via a volume integral, limited at the discrete level to one single layer of FEs touching the boundary, is the natural way to average it as a weak quantity.

At the discrete level, the reference quantity $\boldsymbol{a}_1$ in (19) is initially given in the mesh of the reference problem. It has afterward to be expressed in the mesh of the perturbation problem for being used in (19). This can be done through a projection method [7] of its curl limited to the layer of finite elements touching $\Omega_{cpe,1}$.

C. Perfect magnetic materials perturbed to real ones

When perfect magnetic materials $\Omega_{cpm,1}\subset\Omega_{c,1}\subset\Omega_1$ are considered, the reference formulation is of the form (16) where the perfect materials are extracted from Ω_1 and $\Omega_{c,1}$ and are only involved through their boundaries $\Gamma_{cpm,1}$ added to $\Gamma_{h,1}$ with the BC (12a), i.e.

$$<\boldsymbol{n}\times\boldsymbol{h}_1, \boldsymbol{a}'>_{\Gamma_{cpm,1}\subset\Gamma_{h,1}} = <0, \boldsymbol{a}'>_{\Gamma_{cpm,1}^+} = 0. \quad (20)$$

For the associated perturbation formulation (17), the IC (14) must be strongly expressed via the vector potential $\boldsymbol{a}_2$. It can be expressed with a known discontinuous component $\boldsymbol{b}_{d,2}$ of $\boldsymbol{b}_2$ only acting outside $\Omega_{cpm,2}$, with $\boldsymbol{b}_2=\boldsymbol{b}_{c,2}+\boldsymbol{b}_{d,2}$, i.e.

$$[\boldsymbol{n}\cdot\boldsymbol{b}_2]_{\Gamma_{cpm,2}} = \boldsymbol{n}\cdot\boldsymbol{b}_{d,2}|_{\Gamma_{cpm,2}^+} = -\boldsymbol{n}\cdot\boldsymbol{b}_1|_{\Gamma_{cpm,1}^+}, \quad (21)$$

or with an associated known discontinuous component $\boldsymbol{a}_{d,2}$ of $\boldsymbol{a}_2$, with $\boldsymbol{a}_2=\boldsymbol{a}_{c,2}+\boldsymbol{a}_{d,2}$, also only acting outside $\Omega_{cpm,2}$, i.e.

$$\boldsymbol{a}_{d,2}|_{\Gamma_{cpm,2}^+} = -\boldsymbol{a}_1|_{\Gamma_{cpm,1}^+}. \quad (22)$$

The IC (15) cancels the surface integral term related to $\Gamma_{cpm,2}^+$ in (17), i.e. $<[\boldsymbol{n}\times\boldsymbol{h}_2]_{\Gamma_{cpm,2}}, \boldsymbol{a}'>_{\Gamma_{cpm,2}^+} = <0, \boldsymbol{a}'>_{\Gamma_{cpm,2}^+}$.

V. Application

Two core-inductor systems are considered as test problems (Figs. 1-2). Their five copper stranded inductors are connected in series. The core of system 1 (Fig. 1) is made of aluminium ($\sigma_{Al} = 2.7\ 10^7\ \Omega^{-1}\text{m}^{-1}$). The one of system 2 (Fig. 2) is made of steel ($\sigma_{St} = 10^6\ \Omega^{-1}\text{m}^{-1}$ and relative permeability $\mu_{r,St} = 270$). A 2D model with a vertical symmetry axis is considered. For a direct comparison with the IBC technique, a frequency domain analysis is done. However, the subdomain perturbation technique can be directly applied to time domain analyses without any change. The working frequencies are

5 kHz and 500 Hz for systems 1 and 2 respectively, to give equal skin depths $\delta = \delta_{Al} = \delta_{St} = 1.37$ mm. The core half-width is 12.5 mm (9.1 δ) and its height is 50 mm (36.5 δ). Holes are considered in the core in order to point out the effect of several corners. They are non-uniformly distributed to allow for different lengths of plane portions between them (small lengths should penalize the IBC technique).

The magnetic flux lines are shown in Figs. 1 and 2 for the different calculations performed, i.e. the conventional FE approach, the reference problem and the perturbation problem. Figs. 3 and 4 show the eddy current and Joule power density distributions in the core, as well as the relative error on these quantities made by the IBC versus the subdomain FE approach, the results of which are checked to be very similar to those of the conventional FE approach. The error significantly increases in the vicinity of the conductor corners: it reaches 50% for the Joule power density and 30% for the current density in the smallest plane portions. This affects the total losses accuracy when the size of the conductor portions decreases. The error with the IBC is shown to be significant up to a distance of about 3 δ from each corner. A good accuracy is only obtained beyond this distance for system 1. The IBC error increases with δ with respect to the structure dimensions, whereas the developed approach successfully and accurately adapts its solution to any δ.

VI. Conclusions

The developed subdomain perturbation method offers a way to uncouple FE regions in eddy current frequency and time domain analyses, allowing the solution process to be lightened. The skin and proximity effects in conductors can be accurately determined in a wide frequency range, allowing precise Joule losses calculations.

Once calculated, the reference solution can be used in each subproblem not only for a single signal but for several signals. This allows efficient parameterized analyses on the signal form and the electric and magnetic characteristics of the conductors in a wide range, i.e. on all the parameters affecting the skin depth. Nonlinear analyses, e.g. with temperature dependent conductivities, could then benefit from this.

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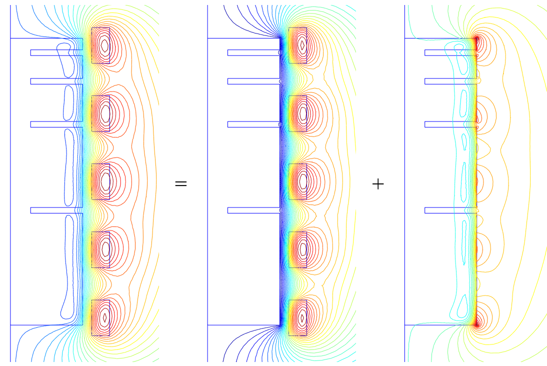

Fig. 1. Magnetic flux lines for the conventional FE solution $\boldsymbol{b}$ (*left*), the reference solution $\boldsymbol{b}_1$ (*middle*) and the perturbation solution $\boldsymbol{b}_2$ (*right*); system 1 with a conductive non-magnetic core.

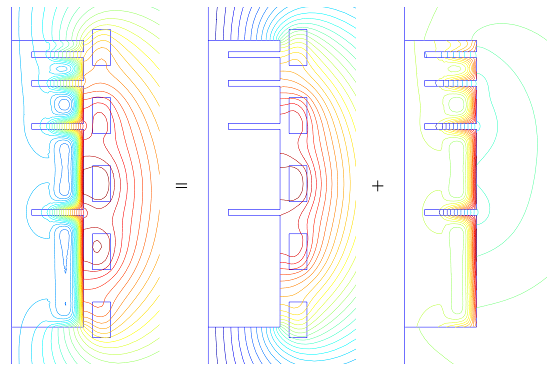

Fig. 2. Magnetic flux lines for $\boldsymbol{b}$ (*left*), $\boldsymbol{b}_1$ (*middle*) and $\boldsymbol{b}_2$ (*right*); system 2 with a conductive magnetic core.

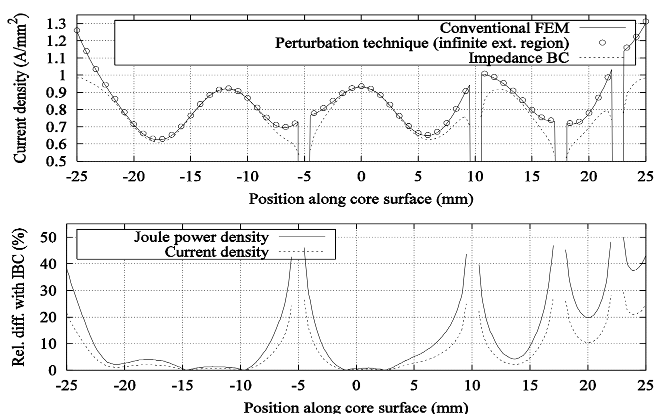

Fig. 3. Eddy current density along the core surface for the conventional FE solution, the perturbation technique and the IBC technique (*top*); relative difference between solutions of the last two techniques (*bottom*); system 1 with a conductive non-magnetic core.

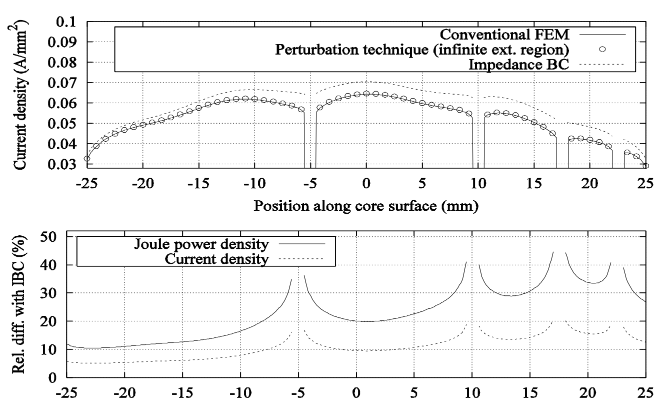

Fig. 4. Same as Fig. 3 but for system 2 with a conductive magnetic core.